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# ON PERIODIC, ALMOST STATIONARY MOTIONS OF A RIGID BODY ABOUT A FIXED POINT

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Periodic, nearly stationary solutions of the problem of motion of a rigid body about a fixed point are obtained by applying the Liapunov method [1] to the system of equations of motion.

The initial system of equations in isothermal coordinates [2]

$$x'' = -\Omega y' + U_x, \quad y'' = \Omega x' + U_y \tag{1}$$

admits the Jacobi integral

$$\begin{gathered} x'^2 + y'^2 = 2U \\ U = \varkappa (h + W), \quad \varkappa = B\,(1 - k^2\sigma^2 - k'^2\rho^2) \\ W = U_0 - \frac{f^2}{2B}\left(1 - \frac{A-B}{A}\,\sigma^2\right)\left(1 + \frac{B-C}{C}\,\rho^2\right), \quad k^2 = 1 - k'^2 = \frac{A-B}{A-C} \end{gathered} \tag{2}$$

Here $\Omega$, $\sigma$ and $\rho$ are specified functions of $x$ and $y$; $U_0$ is the force function; $h$ and $f$ are the energy and area constants.

The stationary solutions are given by the equations

$$U_x = U_y = U = 0 \tag{3}$$

Eliminating the points $|\sigma| = |\rho| = 1$, we can write (3) in the form

$$W_x = 0, \quad W_y = 0, \quad h + W = 0$$

The first two equations define the coordinates of the set of points $\{P_r(x_{0r}(f), \ y_{0r}(f))\}$ corresponding to the stationary motions and the third equation defines, for each of these points, the appropriate value of the Jacobi constant.

We assume that a stationary rotation $P_0(f)$, $h_0(f)$. is given. We shall construct periodic solutions of the system (1) with the value of the parameter $f$ fixed in the neighborhood of the point $P_0$. Let us set $x = x_0 + \xi$, $y = y_0 + \eta$. Equations (1) and (2) will now become

$$\begin{gathered} \xi'' + \Omega(x_0 + \xi, \ y_0 + \eta)\,\eta' - U_\xi(x_0 + \xi, \ y_0 + \eta; \ h) = 0 \quad (4) \\ \eta'' - \Omega(x_0 + \xi, \ y_0 + \eta)\xi' - U_\eta^*(x_0 + \xi, \ y_0 + \eta; \ h) = 0 \\ \xi'^2 + \eta'^2 - 2U(x_0 + \xi, \ y_0 + \eta; \ h) = 0 \end{gathered}$$

where $h$ is the Jacobi constant for the perturbed motion. The zero solution of this system with $h = h_0$ corresponds to the stationary motion specified above.

We construct the periodic solutions of the system (4) using the Liapunov theorem on holomorphic integral [1], and obtain them in the form of series in powers of the parameter $c$, which in the present case depends on $h$. The solution sought becomes zero when $c = 0$ and then $h = h_0$. Let us set

$$\xi = \sum_{s=1}^{\infty} c^s x^{(s)}, \quad \eta = \sum_{s=1}^{\infty} c^s y^{(s)}, \quad h = h_0 + \sum_{s=2}^{\infty} c^s h_s \tag{5}$$

where $h_s$ are constants and $x^{(s)}$, $y^{(s)}$ are $T$-periodic functions of $\tau$ The period can be written as

$$T = \frac{2\pi}{\nu} = \frac{2\pi}{\nu_0}\left(1 + \sum_{s=2}^{\infty} T_s c^s\right) \tag{6}$$

We introduce a new parameter $u = \nu\tau$ and seek the solution in the form

$$x^{(s)} = a_{1s}{}^\circ + \sum_{r=1}^{s} (a_{1s}^{(r)} \cos ru + b_{1s}^{(r)} \sin ru) \tag{7}$$

$$y^{(s)} = a_{2s}{}^\circ + \sum_{r=1}^{s} (a_{2s}^{(r)} \cos ru + b_{2s}^{(r)} \sin ru)$$

The first approximation equations are

$$\nu_0{}^2 \frac{d^2 x^{(1)}}{du^2} + \Omega_0 \nu_0 \frac{dy^{(1)}}{du} + \alpha x^{(1)} + \beta y^{(1)} = 0$$

$$\nu_0{}^2 \frac{d^2 y^{(1)}}{du^2} - \Omega_0 \nu_0 \frac{dx^{(1)}}{du} + \beta x^{(1)} + \gamma y^{(1)} = 0$$

$$(\alpha = -\varkappa_0 W_{xx0}, \ \beta = -\varkappa_0 W_{xy0}, \ \gamma = -\varkappa_0 W_{yy0})$$

where the zero subscript corresponds to the values $x = x_0$, $y = y_0$.
The frequencies are given by

$$\nu_0 = 2^{-1/2} \{\alpha + \gamma + \Omega_0{}^2 \pm [(\alpha + \gamma + \Omega_0{}^2)^2 - 4(\alpha\gamma - \beta^2)]^{1/2}\}^{1/2} \tag{8}$$

Let us consider two possible cases.

1) $\alpha\gamma - \beta^2 > 0$ ($W$ attains an extremal value at $P_0$) in which case we have two different frequencies when the inequality

$$\Omega_0{}^2 > 2\sqrt{\alpha\gamma - \beta^2} - \alpha - \gamma$$

holds (this condition always holds at the maximum points of the function $W$).

2) $\alpha\gamma - \beta^2 < 0$ ($W$ has a saddle point at $P_0$) when we have a single frequency given by the formula (8) with the plus sign.

In the case 1) every frequency corresponds to a single family of periodic motions dependent on the arbitrary parameter $c$, while in the case 2) we have one such family.

The first approximation for the family of periodic solutions corresponding to the frequency $\nu_0$ is obtained in the form

$$x = x_0 + (\nu_0^2 - \gamma)c \sin u$$
$$y = y_0 + c\,(\beta \sin u - \Omega_0\nu_0 \cos u)$$
$$h = h_0 + \frac{c^2\nu_0^2}{2\varkappa_0}[(\nu_0^2 - \gamma)^2 + \beta^2 + \Omega_0^2\gamma]$$

In the first approximation the periodic trajectories represent similar ellipses with a common center $P_0$. The axis of these ellipses form the following angle with the $x$-axis:

$$\frac{1}{2}\,\mathrm{arc\,tg}\,\frac{2\beta(\nu_0^2 - \gamma)}{(\nu_0^2 - \gamma)^2 - \beta^2 - \Omega_0^2\nu_0^2}$$

Each trajectory corresponds to a single value of the Jacobi constant.

In terms of the Euler angles the above motion represents a superposition of nutational oscillations of small amplitude and of small oscillations about the axis of self-rotation, on the perturbed precession with periodically varying velocity.

In the case when the frequencies defined by (8) are incommensurable, all subsequent coefficients appearing in the series (6) and (7) as well as $h_s$ can be found. The series will converge absolutely for every $c$ as long as $|c|$ does not exceed a certain limit. The coordinates are connected with time by means of the following expressions

$$t - t_0 = \int_0^\tau \varkappa\,(x\,(\tau),\, y\,(\tau))\, d\tau$$

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## REFERENCES

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